A MODIFICATION TO THE THEORY OF UNIFORM DISTRIBUTION AND ITS APPLICATION TO THE THEORY OF OSCILLATIONS

PMM Vol. 32, No. 1, 1968, pp. 27-35

M.V. MIRONOV
(Leningrad)

(*Received February 1, 1967*)

The modification of the Weyl's theory of uniform distribution [1 to 5] presented in this paper widens its region of applicability. In particular, we find that the well-known formula which refers to the process of time-averaging of functions depending on several harmonics of differing frequencies and reduces an infinite integral to a repeated one, can now be applied over a much wider region. As an example, we consider biharmonic oscillations of a mechanical system in the presence of dry and viscous friction, and square law resistance.

1. Many problems of the theory of oscillations necessitate the computing of mean values of functions which depend, in general, on a set of harmonics possessing arbitrary frequencies. Such is the case of the Van der Pohl method of averaging when applied to the systems with n degrees of freedom, while another example is given in [6], which presents a method of investigating polyharmonic oscillations in nonlinear systems based on the process of averaging the Lagrangian and the function W, the latter characterizing the nonconservative forces.

Let us consider a real function $f = f(x_0, \ldots, x_r)$ periodic in $x_0, x_1, \ldots, x_r$ with the unit period and strictly Riemann integrable over the region $0 \leqslant x_\nu < 1$ $(\nu = 0, \ldots, r)$. We assume that the function

$$f(t) = f(\omega_0 t, \ldots, \omega_r t) \tag{1.1}$$

obtained from f by replacing x_ν with $\omega_\nu t$ $(\nu = 0, \ldots, r)$ where $\omega_0, \ldots, \omega_r$ are real numbers, has a bound defined by

$$\langle f \rangle = M[f(t)] \doteq \lim_{(\tau - \tau_0) \to \infty} \frac{1}{\tau - \tau_0} \int_{\tau_0}^{\tau} f(t)\, dt \tag{1.2}$$

which we shall call the mean value of $f(t)$. The integral appearing in (1.2) may be difficult to integrate when $r \geqslant 1$, but (1.2) can be replaced by a much simpler formula [2 to 4]

$$\langle f \rangle = \int_0^1 \ldots \int_0^1 f(x_0, \ldots, x_r)\, dx_0 \ldots dx_r \tag{1.3}$$

provided that the numbers $\omega_1/\omega_0, \ldots, \omega_r/\omega_0$ are rationally independent (numbers $\xi_1, \ldots, \xi_r$ are said to be rationally independent if no set of integers $(m_0, m_1, \ldots, m_r) \neq (0, 0, \ldots, 0)$ satisfies the Eqs. $m_1 \xi_1 + \ldots + m_r \xi_r = m_0$). A question now arises, whether a result resembling (1.3) could not be obtained for the case when some of $\omega_0, \ldots, \omega_r$ are commensurable. We shall answer this, using a modification of the Weyl's theory of uniform distribution. This we present below as the theory of **P**-uniform distribution.

2. Definition 2.1. Let the vector $\mathbf{P} = (P_1, \ldots, P_r)$ be an integral vector (i.e. all its projections possessing integral values) with all its projections being positive. Then the system

$$x_\nu(k) \qquad (\nu = 1, \ldots, r, \; k = 1, 2, 3, \ldots) \tag{2.1}$$

of real functions of natural argument shall be $\mathbf{P}$-uniformly distributed ($\mathbf{P}$-u.d.) mod 1, if, for an arbitrary real vector $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_r)$ together with the integral vectors $\mathbf{i} = (i_1, \ldots, i_r)$ and $\mathbf{j} = (j_1, \ldots, j_r)$ whose projections satisfy the inequalities $0 \leqslant i_\nu < j_\nu \leqslant P_\nu$, the relations

$$\mu\left[\chi(\mathbf{i}, \mathbf{j}, \mathbf{P}; \mathbf{x}(k) + \boldsymbol{\sigma})\right] = \prod_{\nu=1}^{r} \frac{j_\nu - i_\nu}{P_\nu} \tag{2.2}$$

$$\mu[\varphi(k)] = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(k), \quad \chi(\mathbf{i}, \mathbf{j}, \mathbf{P}; \mathbf{x}) = \prod_{\nu=1}^{r} \chi(i_\nu, j_\nu, P_\nu; x_\nu) \tag{2.3}$$

hold. Here $\mathbf{x} = (x_1, \ldots, x_r)$,

$$\chi(i_\nu, j_\nu, P_\nu; x_\nu) = \begin{cases} 1, & i_\nu / P_\nu < \{x_\nu\} < j_\nu / P_\nu \\ 1/2, & \{x_\nu\} = i_\nu / P_\nu, \; \{x_\nu\} = j_\nu / P_\nu \\ 0, & \{x_\nu\} < i_\nu / P_\nu, \; \{x_\nu\} > j_\nu / P_\nu \end{cases} \tag{2.4}$$

and $\{x_\nu\}$ is the fractional part of x_ν.

Theorem 2.1. The system of functions (2.1) will be $\mathbf{P}$-u.d. mod 1 if and only if the limit relation

$$\mu[\varphi(\mathbf{x}(k) + \boldsymbol{\sigma})] = \frac{1}{P_1, \ldots, P_r} \sum_{h_r=1}^{P_r} \cdots \sum_{h_1=1}^{P_1} \varphi^*\left(\frac{h_1}{P_1}, \ldots, \frac{h_r}{P_r}\right) \tag{2.5}$$

holds for any function $\phi(\mathbf{x}) = \phi(x_1, \ldots, x_r)$ Riemann integrable over the unit cube $0 \leqslant x_\nu \leqslant 1$ ($\nu = 1, \ldots, r$) and periodic in $x_1, \ldots, x_r$ with the period equal to unity, for which the limit $\mu[\phi\, \mathbf{x}(k) + \sigma)]$ is meaningful. In (2.5) $\phi^*(h_1/P_1, \ldots, h_r/P_r)$ denotes a value of $\phi(\mathbf{x})$ for which the inequality

$$\varphi_i(h_1/P_1, \ldots, h_r/P_r) \leqslant \varphi^*(h_1/P_1, \ldots, h_r/P_r) \leqslant \varphi_s(h_1/P_1, \ldots, h_r/P_r)$$

holds. Here $\phi_i(h_1/P_1, \ldots, h_r/P_r)$ and $\phi_s(h_1/P_1, \ldots, h_r/P_r)$ are the corresponding exact lower and upper bounds of $\phi(\mathbf{x})$ on an open rectangular parallelepiped $(h_\nu - 1)/P_\nu < x_\nu < h_\nu/P_\nu$ ($\nu = 1, \ldots, r$).

Proof. Necessity: Assuming that

$$\varphi(x_1, \ldots, x_r) = 1/2 \;\; [\varphi(x_1 + 0, \ldots, x_r + 0) + \varphi(x_1 - 0, \ldots, x_r - 0)]$$

we find, that each of the functions considered in the above theorem satisfies

$$\varphi(\mathbf{x}(k) + \boldsymbol{\sigma}) = \sum_{h_r=1}^{P_r} \cdots \sum_{h_1=1}^{P_1} \varphi_k^*\left(\frac{h_1}{P_1}, \ldots, \frac{h_r}{P_r}\right) \chi(\mathbf{h} - \mathbf{1}, \mathbf{h}, \mathbf{P}; \mathbf{x}(k) + \boldsymbol{\sigma}) \tag{2.6}$$

where $\mathbf{h} = (h_1, \ldots, h_r)$, $1 = (1, \ldots, 1)$ and

$$\varphi_k^*\left(\frac{h_1}{P_1}, \ldots, \frac{h_r}{P_r}\right) = \begin{cases} \varphi(\ldots, x_\nu(k) + /\sigma_\nu), \ldots), & (h_\nu - 1)/P_\nu < \{x_\nu(k) + /\sigma_\nu)\} < h_\nu / P^\nu \\ \varphi(\ldots, (h_\nu - 1)/P_\nu + 0, \ldots), & \{x_\nu(k) + /\sigma_\nu)\} \leqslant (h_\nu - 1)/P_\nu \\ \varphi(\ldots, h_\nu / P_\nu - 0, \ldots), & \{x_\nu(k) + /\sigma_\nu\}) \geqslant h_\nu / P_\nu \\ & (\nu = 1, \ldots, r) \end{cases}$$

Consequently

$$\varphi_i(h_1/P_1, \ldots, h_r/P_r) \leqslant \varphi_k^*(h_1/P_1, \ldots, h_r/P_r) \leqslant \varphi_s(h_1/P_1, \ldots, h_r/P_r) \tag{2.7}$$

If we now assume that the system of functions $x_\nu(\mathbf{k})$ is $\mathbf{P}$-u.d. mod 1, then, summing both parts of (2.6) over k and taking into account (2.7) and (2.2) we obtain, in the limit, (2.5).

S u f f i c i e n c y : Let (2.5) hold for the system of functions (2.1). Selecting the function $\chi(\mathbf{i}, \mathbf{j}, \mathbf{P}; \mathbf{x})$ as $\phi(\mathbf{x})$ and taking into account the obvious formula

$$\sum_{h_r=1}^{P_r} \cdots \sum_{h_1=1}^{P_1} \chi^* \left(\mathbf{i}, \mathbf{j}, \mathbf{P}; \frac{h_1}{P_1}, \ldots, \frac{h_r}{P_r}\right) = \prod_{\nu=1}^{r} (j_\nu - i_\nu)$$

we arrive at the relation (2.2). Therefore (2.1) is $\mathbf{P}$-u.d. mod 1.

For the system of functions of the form

$$x_\nu(k) = q_\nu(k) / P_\nu \qquad (\nu = 1, \ldots, r, \; k = 1, 2, 3, \ldots) \tag{2.8}$$

where $q_\nu(k)$ are integers we have a sharper theorem, the proof of which is analogous to that of Theorem 2.1. It is

T h e o r e m 2.2. The system of functions (2.8) is $\mathbf{P}$-u.d. mod 1 if and only if the limit relation

$$\mu\,[\varphi(\mathbf{x}(k) + \sigma)] = \frac{1}{P_1, \ldots, P_r} \sum_{h_r=1}^{P_r} \cdots \sum_{h_1=1}^{P_1} \varphi\left(\frac{h_1}{P_1} + \sigma_1, \ldots, \frac{h_r}{P_r} + \sigma_r\right)$$

holds for every $\phi(\mathbf{x})$ given in Theorem 2.1.

T h e o r e m 2.3. The system of functions (2.1) is $\mathbf{P}$-u.d. mod 1, if the limit relation

$$\mu\;[e\,(\mathbf{m} \cdot \mathbf{x}\,(k))] = 0 \tag{2.9}$$

where $e(z) = e^{2\pi i z}$ holds for the integral vectors $\mathbf{m} = (m_1, \ldots, m_r) \neq (0, \ldots, 0)$ whose components are $m_\nu \neq h_\nu P_\nu$, $h_\nu = \pm 1, \pm 2, \ldots$ $(\nu = 1, \ldots, r)$.

P r o o f. By definition, functions $\chi(i_\nu, j_\nu, P_\nu; x_\nu)$ can be represented by convergent Fourier series

$$\chi(i_\nu, j_\nu, P_\nu; x_\nu + \sigma_\nu) = \sum_{m_\nu=-\infty}^{\infty} a_\nu(m_\nu)\, e\,(m_\nu x_\nu) \tag{2.10}$$

$$a_\nu(m_\nu) = \int_0^1 \chi\,(i_\nu, j_\nu, P_\nu; x_\nu + \sigma_\nu)\, e\,(-m_\nu x_\nu)\, dx_\nu$$

and we easily see that

$$a_\nu(0) = (j_\nu - i_\nu) / P_\nu, \qquad a_\nu\,(h_\nu P_\nu) = 0 \qquad (h_\nu = \pm 1, \pm 2, \pm 3, \ldots) \tag{2.11}$$

Let us now consider the system (2.1) assuming that it satisfies the conditions of the theorem. By (2.10) we have, for (2.3)

$$\chi\,(\mathbf{i}, \mathbf{j}, \mathbf{P}; \mathbf{x}\,(k) + \sigma) = \sum_{m_r=-\infty}^{\infty} \cdots \sum_{m_1=-\infty}^{\infty} a_1(m_1) \ldots a_r(m_r)\, e\,(\mathbf{m} \cdot \mathbf{x}\,(k)) \tag{2.12}$$

Performing the summation of both parts of (2.12) over k and taking into account (2.9) and (2.11), we obtain (2.2). Therefore the system of functions (2.1) is $\mathbf{P}$-u.d. mod. 1.

D e f i n i t i o n 2.2. Numbers $y_1, \ldots, y_r$ shall be called $\mathbf{P}$-rationally independent if no set of integers $(m_0, m_1, \ldots, m_r) \neq (0, 0, \ldots, 0)$ such that $m_\nu \neq h_\nu P_\nu$, $h_\nu = \pm 1, \pm 2, \ldots$ $(\nu = 1, \ldots, r)$ satisfies the equations $m_1 y_1 + \ldots + m_r y_r = m_0$.

T h e o r e m 2.4. If the numbers $y_1, \ldots, y_r$ are $\mathbf{P}$-rationally independent, then the system of functions $x_\nu(k) = k y_\nu$ $(\nu = 1, \ldots, r;\; k = 1, 2, \ldots)$ is $\mathbf{P}$-u.d. mod. 1.

P r o o f. A well-known formula for the sum of a geometric progression yields

$$\left|\sum_{k=1}^{n} e\,(\mathbf{m} \cdot \mathbf{x}\,(k))\right| = \left|\sum_{k=1}^{n} e\,(k\eta)\right| = \left|\frac{e\,((n+1)\,\eta) - e\,(\eta)}{1 - e\,(\eta)}\right| \leqslant \frac{2}{|1 - e\,(\eta)|} = \frac{1}{|\sin \pi\eta|}$$

where by conditions of the theorem $\eta = \mathbf{m}\cdot\mathbf{y} = m_1 y_1 + \ldots + m_r y_r$ cannot be an integer if $\mathbf{m} \neq (0,\ldots,0)$ and all $m_\nu \neq h_\nu P_\nu$, $h_\nu = \pm 1, \pm 2, \pm 3, \ldots$. Consequently the relation (2.9) of Theorem 2.3 holds, and this proves the theorem.

Here it should be noted that if we require in Definition 2.1 that the Eqs. (2.2) holds for any, arbitrarily large numbers $P_1,\ldots, P_r$, then the definition will become equivalent to the definition of uniform distribution (u.d.) in the Weyl's sense [3 and 5]. Similarly, the theorems given above will yield the corresponding theorems of the theory of uniform distribution. Further, Definition 2.1 shows that when the system of functions is u.d. mod 1, then it is $\mathbf{P}$-u.d. mod 1. The converse is, generally, not true. Let us for example consider the following system of functions, obviously not u.d. mod 1

$$x_\nu(k) = ku_\nu / v_\nu \qquad (\nu = 1,\ldots, r;\ k = 1, 2, 3,\ldots) \tag{2.13}$$

Here u_ν, $v_\nu \neq 0$ are integers such that $D(u_\nu, v_\nu) = 1$ $(\nu = 1,\ldots, r)$ where $D(u_\nu, v_\nu)$ is the greatest common divisor of u_ν and v_ν.

It can easily be shown that the system (2.13) can always be represented as

$$x_\nu(k) = kq_\nu / cp_\nu \qquad (\nu = 1,\ldots,r) \tag{2.14}$$

where q_ν, c and $p_\nu \neq 0$ are integers such that

$$D(p_\nu, q_\nu) = 1, \quad D(p_\mu, p_\nu) = 1 \qquad (\mu, \nu = 1,\ldots,r,\ \mu \neq \nu)$$

Theorem 2.5. System of functions (2.13) representable by (2.14) is $\mathbf{P}$-u.d. mod 1, where $\mathbf{P} = (p_1,\ldots, p_{r-1}, c_1 p_r)$ and $c_1 p_r = v_r = cp_r/D(c, q_r)$.

Proof. First we shall show that the numbers $q_1/cp_1,\ldots, q_r/cp_r$ are $\mathbf{P}$-rationally independent. Let us consider the values of the sum

$$\eta = m_1 q_1 / cp_1 + \ldots + m_r q_r / cp_r \tag{2.15}$$

if the components of the vector $(m_1,\ldots, m_r) \neq (0,\ldots, 0)$ are $m_\nu = h_\nu P_\nu$, $h_\nu = \pm 1, \pm 2, \pm 3, \ldots$ $(\nu = 1,\ldots, r)$. Let $m_r \neq 0$ and $m_\nu = 0$ $(\nu = 1,\ldots, r-1)$. Then $\eta = m_r q_r/cp_r = m_r u_r/v_r$ and it is obvious that, when $m_r \neq h_r P_r = h_r v_r$, the number η is fractional. If even one of the set of numbers $m_1,\ldots, m_{r-1}$, say $m_1 \neq 0$, then η could be written as

$$\eta = m_1 q_1 / cp_1 + B_1 / cG_1 \tag{2.16}$$

Here $G_1 = p_2,\ldots, p_r$ and $B_1 = m_2 q_2 G_1/p_2 + \ldots + m_r q_r G_1/p_r$ are integers. Multiplying (2.16) by cG we find that $cG\eta = m_1 q_1 G_1/p_1 + B_1$ is a fraction since $D(p_1, q_1) = D(p_1, G_1) = 1$ and $m_1 \neq h_1 p_1$ where h_1 is an integer. Therefore η is also fractional, by Definition 2.2 the numbers $q_1/cp_1,\ldots, q_r/cp_r$ are $\mathbf{P}$-rationally independent and Theorem 2.4 completes the proof.

3. We shall now return to the problem mentioned in Section 1. Let us write Expression (1.2) for the mean value of the function (1.1) as

$$\langle f\rangle = \lim_{n\to\infty} \frac{1}{nT} \sum_{k=1}^{n} \int_{kT}^{(k+1)T} f(\omega_0 t_k, \omega_1 t_k, \ldots, \omega_r t_k)\, dt_k$$

where $T_0 = T > 0$ and $\tau = (n+1)T$. We shall assume, for convenience, that T is equal to one of the following magnitudes $|1/\omega_0|,\ldots, |1/\omega_r|$, say $T = T_0 = |1/\omega_0|$. Putting $t_k = \vartheta + kT_0$ we obtain

$$\langle f\rangle = \mu\left[\frac{1}{T_0}\int_0^{T_0} f\left(\omega_0\vartheta, k\frac{\omega_1}{\omega_0} + \omega_1\vartheta, \ldots, k\frac{\omega_r}{\omega_0} + \omega_r\vartheta\right) d\vartheta\right] = \mu\,[\varphi(ky_1, \ldots, ky_r)]$$

Here $y_\nu = \omega_\nu/\omega_0$ $(\nu = 1,\ldots, r)$ and

$$\varphi(\xi_1, \ldots, \xi_r) = \frac{1}{T_0}\int_0^{T_0} f(\omega_0\vartheta, \xi_1 + \omega_1\vartheta, \ldots, \xi_r + \omega_r\vartheta)\, d\vartheta \tag{3.1}$$

Let the numbers $y_1, \ldots, y_r$ be **P**-rationally independent. Using Theorems 2.4 and 2.1 we arrive at the expression

$$\langle f\rangle = \frac{1}{P_1, \ldots, P_r} \sum_{h_r=1}^{P_r} \cdots \sum_{h_1=1}^{P_1} \varphi^*\left(\frac{h_1}{P_1}, \ldots, \frac{h_r}{P_r}\right)$$

If $P_1, \ldots, P_r$ are sufficiently large to justify the following simplification

$$\frac{1}{P_\nu} \sum_{h_\nu=1}^{P_\nu} \varphi^*\left(\ldots, \frac{h_\nu}{P_\nu}, \ldots\right) = \int_0^1 \varphi(\ldots, \xi_\nu, \ldots)\, d\xi_\nu \tag{3.2}$$

then, using (3.1) and putting $x_0 = \omega_0 \vartheta$ we obtain

$$\langle f\rangle = \int_0^1 \cdots \int_0^1 \left[\int_0^1 f\left(x_0, \xi_1 + \frac{\omega_1}{\omega_0} x_0, \ldots, \xi_r + \frac{\omega_r}{\omega_0} x_0\right) dx_0\right] d\xi_1 \ldots d\xi_r =$$

$$= \int_0^1 \left[\int_0^1 \cdots \int_0^1 f(x_0, \xi_1 + y_1 x_0, \ldots, \xi_r + y_r x_0)\, d\xi_1 \ldots d\xi_r\right] dx_0$$

which, after the substitution $x_\nu = \xi_\nu + y_\nu x_0$ $(\nu = 1, \ldots, r)$ and rearrangement of integrals yields

$$\langle f\rangle = \int_0^1 \cdots \int_0^1 \int_0^1 f(x_0, x_1, \ldots, x_r)\, dx_0\, dx_1, \ldots, dx_r \tag{3.3}$$

It should be noted that the function $\phi(\xi_1, \ldots, \xi_r)$ is much smoother than $f(x_0, x_1, \ldots, x_r)$ from which it is obtained by integration. It follows therefore that the substitution (3.2) will be fully justified for virtually all the functions $f(\omega_0 t, \ldots, \omega_r t)$ appearing in the study of oscillation processes, already when $P_\nu > 3$. In particular, it follows that the mean value of $f(t) = f(\omega_1 t, \omega_2 t)$ can be obtained from

$$\langle f\rangle = \int_0^1 \int_0^1 f(x_1, x_2)\, dx_1\, dx_2 \tag{3.4}$$

provided that the ratio $|\omega_1| / |\omega_2|$ is irrational (in which case (3.4) becomes exact), or, provided that it can be represented as a ratio of two relatively simple numbers P_1 and P_2 at least one of which is large enough to make the substitution of the type (3.2) possible, i.e. if P_1 or $P_2 > 3$.

4. Let us consider the function

$$\Psi_{2i} = |\cos(2\pi\omega_1 t - \varphi_1) + \gamma \cos(2\pi\omega_2 t - \varphi_2)|^{2i+1} \tag{4.1}$$

where $i = 0, 1, 2, \ldots$ and $\omega_1, \omega_2 > 0$. This function is uniform and almost periodic, since the corresponding function $|\cos(2\pi x_1 - \phi_1) + \gamma \cos(2\pi x_2 - \phi_2)|^{2i+1}$ is continuous in both its variables x_1 and x_2. Consequently the function (4.1) has the mean value defined by (1.2) (see e.g. [4]). If the ratio of frequencies ω_1 and ω_2 satisfies the requirements made at the end of the previous paragraph, then by (3.4) we have

$$\langle\Psi_{2i}(\gamma)\rangle = \int_0^1 \int_0^1 |\cos 2\pi x_1 + \gamma \cos 2\pi x_2|^{2i+1}\, dx_1\, dx_2$$

Let us assume that $|\gamma| \leqslant 1$ and calculate the integral

$$B_{2i}(\gamma_2) = \int_0^1 |\cos 2\pi x_1 + \gamma_2|^{2i+1}\, dx_1$$

Here $\gamma_2 = \gamma_2(x_2) = \gamma \cos 2\pi x_2$. Utilising the obvious formulas

$$d|z|/dz = \text{sign } z, \qquad z^{2i+2} \text{ sign } z = z|z|^{2i+1}$$

where z is real, we easily obtain

$$B''_{2(i+1)}(\gamma_2) = (2i+2)(2i+3)B_{2i}(\gamma_2), \quad B_{2i}(0) = \frac{2}{\pi}\frac{2i!!}{(2i+1)!!}, \qquad B'_{2i}(0) = 0 \tag{4.2}$$

where $m!!$ denotes the product of natural numbers not greater than m and all of the same parity as m. The latter yields $B_{2(i+2)}(\gamma_2)$ in terms of $B_{2i}(\gamma_2)$. Let us therefore find $B_0(\gamma_2)$. Simple manipulations yield

$$B_0(\gamma_2) = \frac{2}{\pi}(\gamma_2 \arccos \gamma_2 + \sqrt{1-\gamma_2^2}) = \frac{2}{\pi}\left(1 + \frac{\gamma_2^2}{2} + \ldots + \frac{(2n-3)!!\,\gamma_2^{2n}}{2n!!\,(2n-1)} + \ldots\right) \tag{4.3}$$

which can easily be shown to be convergent for $|\gamma| \leqslant 1$. This is also true for the series which shall be obtained below by integrating (4.3). Using (4.2) we find

$$B_2(\gamma_2) = \frac{4}{3\pi}\left(1 + \frac{9}{2}\gamma_2^2 + \frac{3}{8}\gamma_2^4 + \ldots + 9\frac{(2n-5)!!\,\gamma_2^{2n}}{2n!!\,(2n-1)(2n-3)} + \ldots\right) \tag{4.4}$$

and in the similar manner B_4, $B_6, \ldots$. Integrating (4.3) and (4.4) with respect to x_2, we obtain

$$\langle \Psi_0(|\gamma| \leqslant 1)\rangle = \Theta_0(\gamma) = \frac{4}{\pi^2}[2E(\gamma) - (1-\gamma^2)K(\gamma)] =$$

$$= \frac{2}{\pi}\left(1 + \frac{\gamma^2}{4} + \frac{\gamma^4}{64} + \ldots + \left(\frac{(2n-3)!!}{2n!!}\right)^2 \gamma^{2n} + \ldots\right) \tag{4.5}$$

$$\langle \Psi_2(|\gamma| \leqslant 1)\rangle = \Theta_2(\gamma) = \frac{4}{3\pi}\left(1 + \frac{9}{4}\gamma^2 + \frac{9}{64}\gamma^4 + \ldots + 9\left(\frac{(2n-5)!!}{2n!!}\right)^2 \gamma^{2n} + \ldots\right) \tag{4.6}$$

etc. In (4.5) $K(\gamma)$ and $E(\gamma)$ denote complete elliptic integrals of the first and second kind respectively.

In the similar manner we find for $|\gamma| \geqslant 1$,

$$\langle \Psi_0(|\gamma| \geqslant 1)\rangle = \gamma\Theta_0(1/\gamma), \qquad \langle \Psi_2(|\gamma| \geqslant 1)\rangle = \gamma^3\Theta_2(1/\gamma), \ldots \tag{4.7}$$

5. We have said in Section 1, that the obtained results can find application in the theory of oscillation. Let us consider again the problem already investigated in [6].

We shall consider steady-state oscillations of a linear elastic system. A driving force

$$H(t) = \sum_{i=1}^{m} H_i \sin(2\pi\omega_i t - \psi_i) \qquad (H_i \geqslant 0) \tag{5.1}$$

is applied in the x-direction at the point A of the system. This point can move along the x-axis, and the friction is given by

$$f(x') = -\beta_0 \text{ sign } x' - \beta_1 x' - \beta_2 x'^2 \text{ sign} x' \tag{5.2}$$

where β_0 is the dry friction, while β_1 and β_2 are the coefficients of the viscous friction and the square law resistance respectively.

Let us suppose that out of the natural frequencies Ω_j of the system, two, namely $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$ are resonant. We shall explain how resonant oscillations of one frequency influence the resonant oscillations of the other frequency, assuming that the ratio of ω_1 and ω_2 obeys the constraints listed at the end of Section 3 (some cases when these constraints are not obeyed are discussed in [6]). We seek the steady-state oscillations of the system in the form (q_j are the principal coordinates)

$$q_j = a_j' \sin(2\pi\omega_j t - \varphi_j) \quad (j = 1, 2), \qquad q_j = 0 \quad (j > 2) \tag{5.3}$$

under the same assumptions as those in [6]. This corresponds to translation of the point A according to the law

$$x = \sum_j \alpha_j (A)\, q_j = \sum_{j=1}^{2} a_j \sin(2\pi\omega_j t - \varphi_j) \qquad (5.4)$$

where $a_j = \alpha_j(A)\, a_j'$ and $\alpha_j(A)$ are the values of the coefficients of the form of the natural oscillations of the system with frequencies Ω_j at A.

Utilising the arguments of [6] we obtain equations for the unknown parameters of the solution (5.4)

$$\frac{1}{V_j}\frac{\partial\langle\Phi\rangle}{\partial\varphi_j} = \frac{H_j}{2}\cos(\varphi_j - \psi_j), \qquad \frac{\partial\langle\Phi\rangle}{\partial V_j} = \frac{H_j}{2}\sin(\varphi_j - \psi_j) \qquad (j = 1, 2) \qquad (5.5)$$

Here $V_j = 2\pi\omega_j a_j$ and $\Phi = \Phi(x^{\cdot}) = \beta_0 |x^{\cdot}| + \frac{1}{2}\beta_1 x^{\cdot 2} + \frac{1}{3}\beta_2 |x^{\cdot}|^3$ is the dissipation function corresponding to the frictional forces (5.2) [6]. Taking into account the constraints imposed on the ratio ω_1/ω_2 we can, using Formulas (4.5) to (4.7), easily find its mean value ($\gamma = V_1/V_2$)

$$\langle\Phi(\gamma)\rangle = \beta_0 V_2 \langle\Psi_0(\gamma)\rangle + \tfrac{1}{4}\beta_1 (V_1^2 + V_2^2) + \tfrac{1}{3}\beta_2 V_2^3 \langle\Psi_2(\gamma)\rangle \qquad (5.6)$$

We shall now consider two cases:

a) D r y a n d v i s c o u s f r i c t i o n. When $\beta_2 = 0$ we have, from (5.5),

$$\varphi_j - \psi_j = \pi/2, \qquad \beta_1 V_j + 2\beta_0\, \partial(V_2\langle\Psi_0\rangle)/\partial V_j = H_j \quad (j = 1, 2) \qquad (5.7)$$

This easily yields approximate formulas with at least 10% accuracy, e.g.

$$a_1 = \begin{cases} \dfrac{H_1}{\beta_1 k}\left(1 - b_1 + \dfrac{b_1 H_2^2}{H_1^2 (2 - b_1)^2}\right), & H_2 \leqslant 0.5, \quad b_1 \leqslant 1 \\[2ex] \dfrac{H_1(1 - b_2)}{\beta_1 k_1 (1 - b_2/2)}, & H_2 \geqslant 1.25 H_1, \quad b_2 \leqslant 1 \end{cases} \qquad (5.8)$$

where $b_j = 4\beta_0/\pi H_j$ ($j = 1, 2$) while $k_1 = 2\pi\omega_1$.

If $H_2 = 0$, then $a_1 = a_{10} = (1 - b_1) H_1/\beta_1 k_1$ ($b_1 < 1$). Comparing a_1 with a_{10} we find

$$\frac{a_1}{a_{10}} = \begin{cases} 1 + \dfrac{b_1 h^2}{(1 - b_1)(2 - b_1)^2} \geqslant 1, & 0 \leqslant h \leqslant 0.5 \\[2ex] \dfrac{h - b_1}{(h - b_1/2)(1 - b_1)^2} > 1, & h \geqslant 1.25 \end{cases} \qquad (5.9)$$

$$(h = H_2/H_1, \; b_1 < 1)$$

Fig. 1 gives a_1/a_{10} versus $h = H_2/H_1$ obtained directly from (5.7) for various values of the parameter b_1. The points plotted on the curve $b_1 = 0.8$ correspond to Formulas (5.9).

Analysing the above results together with those of [6] we can infer that, in the system with dry friction, the amplitude of the resonant oscillations increases with the appearance of an additional signal of another frequency. Moreover we see from (5.8) that when $H_2 \geqslant 1.25 H_1$, which corresponds approximately to $V_2 \geqslant 1.15\, V_1$, then the resonant amplitude a_1 is a linear function of the amplitude of H_1. Consequently the harmonic possessing the larger velocity amplitude linearizes the dry friction for the "slower" harmonic (cf. [6 and 7]).

b) S q u a r e l a w r e s i s t a n c e. When β_0, $\beta_1 = 0$, Eqs. (5.5) become

$$\varphi_j - \psi_j = \pi/2, \qquad \tfrac{2}{3}\beta_2\, \partial(V_2^3\langle\Psi_2\rangle)/\partial V_j = H_j \quad (j = 1,2) \qquad (5.10)$$

which in turn yield, with an error not exceeding 3%,

$$a_1 = \begin{cases} \sqrt{3\pi H_1/8\beta_2 k_1^2}\,(1 - H_2^2/6H_1^2), & H_2 \leqslant 0.8 H_1 \\ H_1 \sqrt{\pi/6\beta_2 H_2 k_1^2}\,(1 + H_1^2/6H_2^2), & H_2 \geqslant H_1 \end{cases} \qquad (5.11)$$

When $H_2 = 0$, $a_1 = a_{10} = (3\pi H_1/8\beta_2 k_1^2)^{1/2}$. Consequently

$$\frac{a_1}{a_{10}} = \begin{cases} 1 - h^3/6, & 0 < h < 0.8 \\ (2 + 1/_3h^3)/3\sqrt{h}, & h \geqslant 1 \end{cases} \qquad (h = H_2 \,/\, H_1) \qquad (5.12)$$

Fig. 2 shows a_1/a_{10} versus $h = H_2/H_1$. The curve is obtained directly from (5.10), while the points on it are computed according to (5.12). The results obtained imply that, in the system with the square law resistance the amplitude of resonant oscillations decreases with the appearance of another signal of a different frequency. Further, when $H_2 \geqslant 1.5\,H_1$ i.e. when $V_2 \geqslant 2\,V_1$, we can neglect the bracket in (5.11) with little error resulting, and this shows that the square law resistance is practically linearized by the action of the "fast" harmonic on the "slow" one.

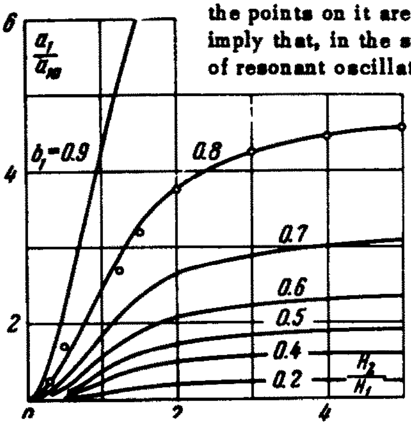

Fig. 1

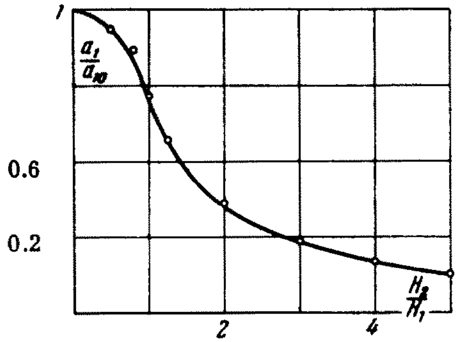

Fig. 2

BIBLIOGRAPHY

1. Weyl, H., Über die Gleichverteilung der Zahlen mod. Eins, Math. Ann., Bd. 77, s.s. 313-352.
2. Polya, G. and Szego, G., Problems and Theorems in Analysis, Vol. 1, Gostekhizdat, 1956.
3. Corput, J. G. Van der. Diophantische Unglechungen, 1. Zur Gleichverteilung Modulo Eins. Acta Math., Bd. 56, p. 373-456, 1931.
4. Levitan, B.M., Almost-periodic Functions, Gostekhizdat, 1953.
5. Cassels, J.W.S., Introduction to Diophantine approximations. Cambridge University Press, 1957.
6. Mironov, M.V., Use of the Hamilton-Ostrogradskii principle in problems of the theory of nonlinear oscillations. PMM, Vol. 31, No. 6, 1967.
7. Kolovskii, M.Z., Influence of high-frequency perturbations on the resonant oscillations in nonlinear systems. Tr. Leningr. politekhn. Inst. im. M.I. Kalinina, No. 226, 1963.

Translated by L.K.